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# Approximate Optimal Planar Guidance of an Aeroassisted Space Vehicle

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**The objective is to develop a guidance law for the lift force during a planar aeroassisted maneuver such that the vehicle is guided from initial entry conditions to specified terminal conditions while minimizing the velocity loss. An approximate solution is proposed to this two-point boundary-value problem, based on the assumption that along the atmospheric portion of typical aeroassisted maneuvers the aerodynamic forces are dominant. The suboptimal control law is thus derived for an approximate model of the vehicle's dynamics and is expressed as a series expansion with respect to a small parameter that reflects the ratio between the gravity and the aerodynamic forces. Zero- and first-order terms of the series are developed for two types of planar maneuvers. The control law is formulated in a feedback form that includes analytic functions and quadrature integrals and thus is suitable for onboard implementation. Numerical examples compare the performance of the exact, zero-order, and first-order solutions. It is shown that the approximate solutions are quite close to the exact one. The first-order control law improves slightly the velocity loss, but generally the zero-order controller is adequate.**

## Introduction

AEROASSISTED maneuvers have been recognized as a potential fuel-saving technique to obtain certain orbital changes.[1] To take full advantage of this potential, an optimal guidance law is required. Such a guidance law has to minimize the fuel consumption by minimizing the magnitude of the velocity impulses that are part of the aeroassisted maneuver. Derivation of the guidance law requires a solution of an optimal control problem, which often requires a numerical solution of a nonlinear two-point boundary-value problem. For an on-line application, one requires a simple guidance law that does not require complex iterations and is expressed in a feedback form.

Much past research was devoted to the derivation of approximate analytic solutions to this optimal control problem. The basic approximation is performed by treating only the aerodynamic forces and neglecting completely the inertial and gravitational forces.[2] This basic approximation is termed the zero-order solution. It is logical and reasonable because by definition an effective aeroassisted maneuver should take place at an altitude range where the aerodynamic forces are dominant. However, to increase the altitude range for which the solution is valid, it is necessary to introduce a higher-order solution that takes into account the inertial and gravitational terms. In Ref. 3, the difference between the gravitational and inertial forces in the vertical direction (also known as Loh's term) was assumed to be constant along the trajectory, thus enabling the integration of the equations of motion, together with the costate equations.

Another method to approximate the inertial and gravity terms, while considering their changing nature, is based on regular perturbations.[4] This method employs a series expansion of the control variable (namely, the lift coefficient) with respect to a small quantity $\varepsilon$, which is essentially the ratio between the atmospheric scale height and the radius of the Earth. This approximation was implemented in Refs. 5 and 6 for an orbital plane change maneuver. This method requires an analytic solution to the zero-order problem. The zero-order problem is obtained by neglecting the gravity and inertial forces in the equations of motion. For many practical maneuvers, an analytic optimal solution can be derived for this zero-order problem. The zero-order solution is then used to obtain the higher-order terms in the control expansion. In Refs. 5 and 6, zero- and first-order terms were developed for the optimal plane change maneuver.

An additional family of approximate solutions is based on matched asymptotic expansion (MAE). In this method, an inner approximate solution is derived for the low-altitude portion, where the aerodynamic forces dominate, and an outer solution is derived for the high-altitude portion, where gravity and inertia dominate. By matching these asymptotic solutions, a composite solution is achieved, which is uniformly valid throughout the altitude range. Reference 7 derives a zero-order MAE solution of a planar maneuver. The solution is expressed in terms of elliptic integrals and requires several iterations to obtain the control law. References 8 and 9 apply the MAE method to the three-dimensional maneuver. The zero-order solution that is derived in Ref. 8 requires the solution of a large number of implicit algebraic equations (20 equations in Ref. 8, later reduced to 6 in Ref. 9) that must be solved at each time step. Reference 8 also recommends that a first-order solution be solved to improve the guidance law. In Ref. 10, a procedure for the first-order solution to the plane change maneuver is formulated but not completely solved.

Comparing the MAE method to the regular expansion method, it seems that although the MAE method provides a uniformly valid solution, the regular perturbation method requires less computational efforts for the zero-order solution, which is essential for the development of the higher-order solutions. Because the aeroassisted maneuver is most effective in the inner portion of the trajectory, it is reasonable to derive a guidance law that is valid for this region up to first order. The advantage is that an approximate first-order feedback guidance with reasonable computational requirements is obtained.

This paper develops approximate optimal guidance laws for a family of aeroassisted planar maneuvers. The approximate solution is based on the regular perturbation method that was discussed in Ref. 4 and later developed in Ref. 5 for an orbital plane change maneuver. Here, the zero- and first-order solutions for an optimal controller of several planar maneuvers are developed. The control law is expressed in a feedback form. The zero-order controller terms are almost analytical and require the solution of only one algebraic equation (polynomial) for one unknown. The first-order terms are obtained by numerical integration of functions of the state variables

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along the already found zero-order solution. The resulting guidance law does not require iterations and thus is suitable for onboard applications. This method is used to compute the control law for two planar maneuvers: a pull-up maneuver and guidance to an optimal cruise altitude.

In the next section, the aeroassisted planar maneuvers that are the subject of this paper are described. Next, the optimization problem is formulated, and zero- and first-order suboptimal control laws are derived. Numerical examples demonstrate the performance of each control law and compare it with the numerical solution of the exact two-point boundary-value problem. Then the advantages of the proposed solutions are summarized. The Appendix discusses several numerical aspects of the solution.

## Aeroassisted Maneuver

A typical aeroassisted maneuver consists of a propulsive deorbit impulse that lowers the perigee of the space vehicle into the atmosphere, an aerodynamic maneuver during the atmospheric passage, and one or more propulsive impulses that bring the vehicle to its final trajectory. In the present work, we are interested in a planar aeroassisted maneuver, in which the initial and the final orbits are coplanar and have the same semimajor axes and same eccentricities but different lines of apsides. The rotation of the line of apsides may be desired, for example, when a local change of the orbital altitude is required, without changing the orbital period. To perform a planar rotation of the line of apsides, a lift force should be applied in the vicinity of the perigee, which is set to be inside the upper layer of the atmosphere. The objective is to develop an onboard guidance law for the lift force that minimizes the velocity loss during the atmospheric passage.

Although a complete optimal guidance law should include the deorbit and reorbit impulses, we deal here only with the atmospheric portion of the maneuver. The vehicle is assumed to be in an initial entry condition set by the deorbit impulse. Typical initial altitude is between 70 and 100 km. At this altitude range, the density is sufficiently large to allow an aerodynamic maneuver. The typical initial velocity range at this altitude for an elliptic orbit is 8–10 km/s (those velocities are greater than the circular velocity). The flight-path angle is of a few degrees (pointing down). The goal is to minimize the velocity loss during the atmospheric passage.

Two types of maneuvers are analyzed in this paper (Fig. 1): 1) a pull-up maneuver, where the final altitude is specified (in this maneuver, the rotation direction of the line of apsides is opposite to the orbital motion of the vehicle), and 2) guidance to an optimal cruise altitude, where the final conditions are set to be the starting conditions of a cruise trajectory at a constant altitude, with the maximum possible lift-to-drag ratio $L/D$. The requirement for a maximum $L/D$ at a constant altitude cruise determines a relationship between the final altitude and the final velocity, i.e., a terminal constraint. In this maneuver, the rotation direction of the line of apsides is the same as the orbital motion of the vehicle.

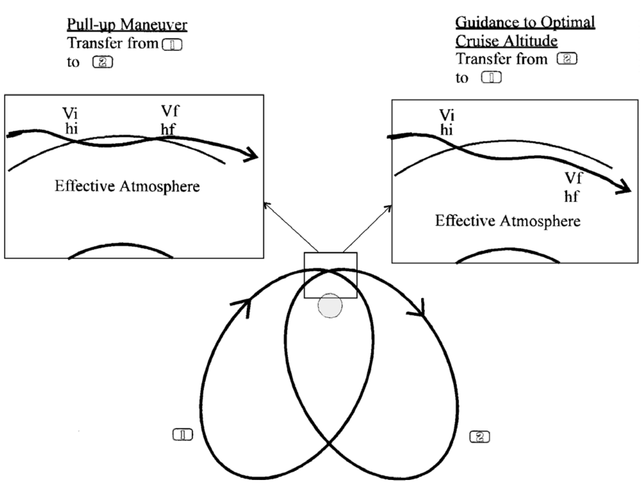


**Fig. 1 Planar aeroassisted maneuvers.**

## Equations of Motion

It is assumed here that a space vehicle initiates the aeroassisted maneuver at a given altitude $h_i$, velocity $V_i$, and flight-path angle $\gamma_i$. For a planar maneuver about a nonrotating planet, the equations of motion of the maneuver are

$$\begin{aligned} \frac{\mathrm{d}\ell}{\mathrm{d}t} &= V\cos\gamma, && \ell(0)=0 \\ \frac{\mathrm{d}V}{\mathrm{d}t} &= -\frac{1}{2}\rho V^2 S\frac{C_D}{m} - g\sin\gamma, && V(0)=V_i \\ \frac{\mathrm{d}\gamma}{\mathrm{d}t} &= \frac{1}{2}\rho V S\frac{C_L}{m} + \left(\frac{V}{r}-\frac{g}{V}\right)\cos\gamma, && \gamma(0)=\gamma_i \\ \frac{\mathrm{d}h}{\mathrm{d}t} &= V\sin\gamma, && h(0)=h_i \end{aligned} \tag{1}$$

The state variables are $\ell$ (arclength), $V$ (velocity), $\gamma$ (flight-path angle), and $h$ (altitude); $m$ is the mass of the space vehicle, $S$ is the reference area, $\rho$ is the atmospheric density, $r$ is the distance from the center of the planet, and $C_L$ and $C_D$ are the lift and drag coefficients, respectively. The vehicle aerodynamics are modeled as

$$C_D = C_{D_0} + KC_L^2, \qquad |C_L| < C_{L\max}$$

The atmospheric density is assumed to follow the exponential differential relationship

$$\mathrm{d}\rho = -\beta\rho\,\mathrm{d}h$$

where $1/\beta$ is the atmospheric scale height, assumed to be constant over the relevant altitude range. In addition, we assume that the flight-path angle is small, hence $\sin\gamma \cong \gamma$ and $\cos\gamma \cong 1$.

Next, the following nondimensional variables are introduced: 1) nondimensional arclength $\hat{\ell} = \ell/R_e$, where $R_e$ is the radius of the planet; 2) nondimensional velocity $u = \ln[V/\sqrt{(\mu/R_e)}]$, where $\mu$ is the gravitation constant of the planet; 3) nondimensional density $\delta = \rho/\rho_S = e^{-\beta h}$, which replaces the altitude $h$; 4) nondimensional control variable $\eta = C_L/C_L^*$, where $C_L^*$ is the lift coefficient for which the lift-to-drag ratio of the vehicle is maximum [for the parabolic aerodynamic model, $C_L^* = \sqrt{(C_{D_0}/K)}$]; and 5) nondimensional independent variable $\mathrm{d}z = -\mathrm{d}\delta/\gamma$, $z(0) = 0$. Substituting $\delta = \rho/\rho_S = e^{-\beta h}$, we get

$$\begin{aligned} \mathrm{d}z &= -\mathrm{d}\delta/\gamma = -(1/\gamma)\,\mathrm{d}(e^{-\beta h}) = (\beta/\gamma)e^{-\beta h}\,\mathrm{d}h \\ &= (\beta\delta/\gamma)\,\mathrm{d}h = \beta\delta V(\sin\gamma/\gamma)\,\mathrm{d}t \end{aligned}$$

Because $\beta\delta V > 0$, then for $-\pi < \gamma < \pi$ we have $\mathrm{sign}(\mathrm{d}z) = \mathrm{sign}(\mathrm{d}t)$. For small flight-path angles, $\sin\gamma/\gamma \approx 1$. Then $\mathrm{d}z = \beta\delta V\,\mathrm{d}t$ and is proportional to $\mathrm{d}t$. Therefore $z$ is monotonous in time and can replace the time as the independent variable.

The equations of motion are then expressed in terms of the nondimensional variables as

$$\frac{\mathrm{d}\hat{\ell}}{\mathrm{d}z} = \frac{\varepsilon}{\delta}, \qquad \hat{\ell}(0) = 0 \tag{2a}$$

$$\frac{\mathrm{d}u}{\mathrm{d}z} = -A_1(1+\eta^2) - \varepsilon\gamma\frac{e^{-2u}}{\delta}, \qquad u(0) = u_i \tag{2b}$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}z} = A_2\eta + \varepsilon\frac{1-\varepsilon^{-2u}}{\delta}, \qquad \gamma(0) = \gamma_i \tag{2c}$$

$$\frac{\mathrm{d}\delta}{\mathrm{d}z} = -\gamma, \qquad \delta(0) = \delta_i \tag{2d}$$

The terms $A_1$ and $A_2$ are constants defined by

$$A_1 = \frac{\rho_s C_{D_0} S}{2m\beta}, \qquad A_2 = \frac{\rho_s C_L^* S}{2m\beta}$$

$\varepsilon$ is the ratio between the atmospheric scale height $1/\beta$, and the radius of the planet $R_e$ and its value for the Earth is about 0.001.

Because the equations of motion (2) and the optimality objective do not depend explicitly on the arclength variable $\mathcal{L}$, it will not be considered in the optimization process, and thus Eq. (2a) will not be treated in the sequel.

Equations (2) can be written in the matrix form as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}z} = \boldsymbol{f}(\boldsymbol{x}, \eta) + \varepsilon \boldsymbol{g}(\boldsymbol{x}) \tag{3}$$

where $\boldsymbol{x}^T = (u, \gamma, \delta)$ is the state vector and $\boldsymbol{f}(\boldsymbol{x}, \eta)$ and $\boldsymbol{g}(\boldsymbol{x})$ are the vector functions that construct the right-hand sides of Eqs. (2).

## Optimization Problem

The optimization problem is formulated as follows. For given initial conditions and for specified final conditions, find the control law $\eta(u, \gamma, \delta)$ that maximizes the exit (nondimensional) velocity $u_f$, subject to the system dynamics [Eqs. (2)]. Thus, the cost function of the problem is

$$J = -u_f$$

The Hamiltonian of this optimization problem is

$$H = \lambda_u\left[-A_1(1+\eta^2) - \varepsilon\gamma\frac{e^{-2u}}{\delta}\right] + \lambda_\gamma\left[A_2\eta + \varepsilon\frac{1-e^{-2u}}{\delta}\right] - \lambda_\delta\gamma$$

The vector $(\lambda_u, \lambda_\gamma, \lambda_\delta)^T$ contains the Lagrange multipliers and satisfies the Euler–Lagrange equations:

$$\frac{\mathrm{d}\lambda_u}{\mathrm{d}z} = -\frac{\partial H}{\partial u} = -2\lambda_u\varepsilon\gamma\frac{e^{-2u}}{\delta} - 2\lambda_\gamma\varepsilon\frac{e^{-2u}}{\delta}$$

$$\frac{\mathrm{d}\lambda_\gamma}{\mathrm{d}z} = -\frac{\partial H}{\partial \gamma} = \lambda_u\varepsilon\frac{e^{-2u}}{\delta} + \lambda_\delta \tag{4}$$

$$\frac{\mathrm{d}\lambda_\delta}{\mathrm{d}z} = -\frac{\partial H}{\partial \delta} = -\lambda_u\varepsilon\frac{e^{-2u}}{\delta^2} + \lambda_\gamma\varepsilon\frac{1-e^{-2u}}{\delta^2}$$

Because both the Hamiltonian and the equations of motion do not depend explicitly on the independent variable $z$ and because the final value of $z$ is unspecified, we have

$$H = \text{const} = 0 \tag{5}$$

The optimal control law $\eta(u, \gamma, \delta)$ satisfies the optimality condition

$$H_\eta = 0 \Rightarrow \eta^{\text{opt}}(u, \gamma, \delta) = (A_2/2A_1)(\lambda_\gamma/\lambda_u) \tag{6}$$

The boundary conditions of the Euler–Lagrange equations (4) depend on the terminal constraints of each maneuver. Two basic planar maneuvers are considered here.

1) *Pull-up maneuver*: In this maneuver, the vehicle exits the atmosphere at a specified altitude $\delta_f$ and at an unspecified time, velocity, and flight-path angle. The boundary conditions are then

$$\delta(z_f) = \delta_f, \qquad \lambda_u(z_f) = \frac{\partial J}{\partial u_f} = -1, \qquad \lambda_\gamma(z_f) = \frac{\partial J}{\partial \gamma_f} = 0 \tag{7a}$$

2) *Guidance toward an optimal cruise altitude*: The vehicle is guided to reach a starting point of a cruise trajectory inside the atmosphere. The terminal constraints for the cruise condition are

$$\gamma(z_f) = 0, \qquad \left.\frac{\mathrm{d}\gamma}{\mathrm{d}z}\right|_{z_f} = 0 \tag{7b}$$

In addition, it is required that the final altitude and velocity are such that the cruise trajectory starts at the maximum lift-to-drag ratio. Because the vehicle is arriving from an elliptical orbit, the velocity of the vehicle is larger than the circular velocity at the cruise altitude. Hence a negative (pointing down) lift is required. The additional terminal constraint is then $\eta(z_f) = -1$. The combined terminal conditions ensure that the vehicle is guided to a cruise altitude at which the lift-to-drag ratio is maximum. The terminal altitude $\delta_f$ is not specified but rather is determined from the other terminal conditions.

To express the optimization problem in the standard Mayer form, the terminal conditions $(\mathrm{d}\gamma/\mathrm{d}z)|_f = 0$ and $\eta_f = -1$ are used to obtain a terminal constraint on the states. Using Eq. (2c) with $(\mathrm{d}\gamma/\mathrm{d}z)|_f = 0$ and $\eta_f = -1$, we obtain

$$-A_2 + \varepsilon\frac{1-e^{-2u_f}}{\delta_f} = 0 \tag{7c}$$

The second maneuver thus has two terminal constraints: Eq. (7c) and $\gamma(z_f) = 0$.

The complete equation system for the optimization problem includes the equations of motion (2), the constant Hamiltonian (5), the optimality condition (6), and the Euler–Lagrange equations (4) with the appropriate boundary conditions (7a) or (7c). This forms a two-point boundary-value problem. Exact solution is tedious, requires numerical integration and iterations, and is difficult for onboard implementation. In the sequel, approximate solutions are derived.

## Approximate Solutions

It has been observed that along a significant portion of an aero-assisted maneuver, the aerodynamic force is much larger than the gravity and inertial forces. Regarding the nondimensional equations (2), we see that the gravity terms are multiplied by a small factor $\varepsilon$. The control law is approximated then as a series expansion of $\varepsilon$:

$$\eta(u, \gamma, \delta) = \sum_{k=0}^{\infty}\eta_k(u, \gamma, \delta)\varepsilon^k \tag{8}$$

Based on the method explained in Refs. 3 and 4, the terms in the series expansion are solved successively. The first term $\eta_0(u, \gamma, \delta)$ is the zero-order solution and is determined by solving the optimization problem with $\varepsilon = 0$ in the equations of motion (2). To use this method, an analytic solution of the zero-order problem is required. In the following subsections, the zero-order and the first-order solutions are derived. These solutions lead to a feedback control law, i.e., the controller is expressed as a function of the state variables.

### Zero-Order Solution

The first term in the series expansion (8), $\eta_0(u, \gamma, \delta)$, is the zero-order solution and is determined by solving the optimization problem with $\varepsilon = 0$ in Eqs. (2).

The zero-order equations of motion are

$$\frac{\mathrm{d}u}{\mathrm{d}z} = -A_1(1+\eta^2), \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}z} = A_2\eta, \qquad \frac{\mathrm{d}\delta}{\mathrm{d}z} = -\gamma \tag{9}$$

The initial conditions at $z = 0$ are $u(0) = u_i$, $\gamma(0) = \gamma_i$, and $\delta(0) = \delta_i$.

The zero-order Hamiltonian is

$$H = -\lambda_{0_u}A_1\left(1+\eta_0^2\right) + \lambda_{0_\gamma}A_2\eta_0 - \lambda_{0_\delta}\gamma \tag{10}$$

where $\lambda_{0_u}$, $\lambda_{0_\gamma}$, and $\lambda_{0_\delta}$ are the zero-order Lagrange multipliers.

From the optimality condition [Eq. (6)], the optimal controller is

$$\eta_0^{\text{opt}} = \frac{A_2\lambda_{0_\gamma}}{2A_1\lambda_{0_u}} \tag{11}$$

The Euler–Lagrange equations are

$$\frac{\mathrm{d}\lambda_{0_u}}{\mathrm{d}z} = -\frac{\partial H}{\partial u} = 0 \tag{12a}$$

$$\frac{\mathrm{d}\lambda_{0_\gamma}}{\mathrm{d}z} = -\frac{\partial H}{\partial \gamma} = \lambda_{0_\delta} \tag{12b}$$

$$\frac{\mathrm{d}\lambda_{0_\delta}}{\mathrm{d}z} = -\frac{\partial H}{\partial \delta} = 0 \tag{12c}$$

Equations (9), (11), and (12), together with their appropriate boundary conditions (7a) or (7c), constitute a two-point boundary-value problem. Solution of these equations provides the optimal controller $\eta_0^{\text{opt}}(z)$ and the optimal trajectory $u^{(0)}(z)$, $\gamma^{(0)}(z)$, and $\delta^{(0)}(z)$ for the zero-order problem. In this case, an analytic solution can be obtained. By integration of Eqs. (12a) and (12c), we get

$$\lambda_{0_u} = \text{const}, \qquad \lambda_{0_\delta} = \text{const}$$

Combining Eqs. (9), (11), and (12b), we get a reduced set of differential equations:

$$\frac{\mathrm{d}u}{\mathrm{d}z} = -A_1\left(1 + \frac{A_2^2\lambda_{0_\gamma}^2}{4A_1^2\lambda_{0_u}^2}\right), \qquad \frac{\mathrm{d}\gamma}{\mathrm{d}z} = \frac{A_2^2\lambda_{0_\gamma}}{2A_1\lambda_{0_u}}$$
$$\frac{\mathrm{d}\delta}{\mathrm{d}z} = -\gamma, \qquad \frac{\mathrm{d}\lambda_{0_\gamma}}{\mathrm{d}z} = \lambda_{0_\delta} \tag{13}$$

The solution is obtained by direct integration of the equations, subject to the initial conditions $u(0) = u_i$, $\gamma(0) = \gamma_i$, and $\delta(0) = \delta_i$:

$$u^{(0)}(z) = -A_1\left[z + \frac{A_2^2}{4A_1^2}\left(\frac{\lambda_{0_\delta}^2}{\lambda_{0_u}^2}\frac{z^3}{3} + \frac{\lambda_{0_\delta}}{\lambda_{0_u}}\frac{C}{\lambda_{0_u}}z^2 + \frac{C^2}{\lambda_{0_u}^2}z\right)\right] + u_i \tag{14a}$$

$$\gamma^{(0)}(z) = \frac{A_2^2}{2A_1}\left(\frac{\lambda_{0_\delta}}{\lambda_{0_u}}\frac{z^2}{2} + \frac{C}{\lambda_{0_u}}z\right) + \gamma_i \tag{14b}$$

$$\delta^{(0)}(z) = -\frac{A_2^2}{2A_1}\left(\frac{\lambda_{0_\delta}}{\lambda_{0_u}}\frac{z^3}{6} + \frac{C}{\lambda_{0_u}}\frac{z^2}{2}\right) - \gamma_i z + \delta_i \tag{14c}$$

$$\lambda_{0_\gamma}(z) = \lambda_{0_\delta}z + C \tag{14d}$$

The optimal controller of Eq. (11) is expressed as a function of the independent variable $z$ by using Eq. (14d):

$$\eta_0^{\mathrm{opt}}(z) = (A_2/2A_1)\left[\left(\lambda_{0_\delta}/\lambda_{0_u}\right)z + \left(C/\lambda_{0_u}\right)\right] \tag{15}$$

The constants $\lambda_{0_\delta}/\lambda_{0_u}$, $C/\lambda_{0_u}$, and $z_f$ are determined by applying the terminal conditions for each of the two maneuvers.

*Pull-Up Maneuver*

For the pull-up maneuver, the boundary conditions are the initial conditions $u_i$, $\gamma_i$, and $\delta_i$ and the terminal conditions (7a).

The constant $C/\lambda_{0_u}$ is obtained from the boundary condition $\lambda_{0_\gamma}(z_f) = 0$. Substituting into Eq. (14d), we get

$$C/\lambda_{0_u} = -\left(\lambda_{0_\delta}/\lambda_{0_u}\right)z_f \tag{16a}$$

Using this result and substituting $\delta(z_f) = \delta_f$ in Eq. (14c), we obtain the constant $\lambda_{0_\delta}/\lambda_{0_u}$:

$$\frac{\lambda_{0_\delta}}{\lambda_{0_u}} = \frac{6A_1}{A_2^2}\left(\frac{\delta_f - \delta_i + \gamma_i z_f}{z_f^3}\right) \tag{16b}$$

By substituting Eqs. (16a) and (16b) into the optimal controller (15), we obtain the open-loop optimal control law $\eta_0^{\mathrm{opt}}(z)$:

$$\eta_0^{\mathrm{opt}}(z) = \frac{3}{A_2}\frac{(z - z_f)(\delta_f - \delta_i + \gamma_i z_f)}{z_f^3} \tag{17a}$$

Because Eq. (17a) is an approximate control law, and because the basic dynamics (1) have modeling errors and do not account for disturbances, we should expect a buildup of deviations from the exact (yet unknown) optimal trajectory. Therefore we desire a closed-loop control law that takes into account the actual states. To get this feedback control law, the current states are treated as new initial states and the controller is recomputed for these new initial conditions and initial independent variables. Therefore the optimal zero-order controller in a feedback form is obtained from Eq. (17a) by dropping the index $i$ from the initial states and substituting $z = 0$:

$$\eta_0^{\mathrm{opt}}(\gamma, \delta) = -\frac{3(\delta_f - \delta + \gamma z_f)}{A_2 z_f^2} \tag{17b}$$

The control law is a function of a single unknown parameter $z_f$, which is a function of the current state. To find $z_f$, we use the condition $H = 0$. By substituting Eq. (17b) into Eq. (10), using Eqs. (16b) and (14b), and setting $H = 0$, we get an equation for the unknown $z_f$. This equation is expressed as a fourth-order polynomial:

$$z_f^4 - \left(3/A_2^2\right)\left[(\gamma z_f)^2 + 4(\delta_f - \delta)\gamma z_f + 3(\delta_f - \delta)^2\right] = 0 \tag{18}$$

Here too the index $i$ of the state variables was dropped, and for each point $u$, $\gamma$, $\delta$ on the trajectory, Eq. (18) is resolved for $z_f$. According to Descartes' rule of signs, there is at least one positive real root to this equation (see further discussion in the Appendix). By substituting the resulting $z_f$ into the controller Eq. (17b), we obtain the optimal feedback control law as a function of the current states $\gamma$, $\delta$. Because $z_f$ is changing slowly, Eq. (18) is solved for $z_f$ only once, at the beginning of the trajectory, and the successive calculations that are performed at each time step are approximated by a single Newton–Raphson iteration:

$$z_{f_{n+1}} = z_{f_n} - \frac{F\left(z_{f_n}\right)}{F'\left(z_{f_n}\right)}$$

where $F(z_{f_n})$ is the left-hand side of Eq. (18) and $F'(z_{f_n})$ is its derivative:

$$F'\left(z_{f_n}\right) = 4z_{f_n}^3 - \left(3/A_2^2\right)\left[2\gamma^2 z_{f_n} + 4(\delta_f - \delta)\gamma\right]$$

The mathematical and numerical aspects of this method are discussed in the Appendix.

As the trajectory approaches the terminal point, $z_f$ (which is proportional to the time-to-go) approaches zero. The control law as formulated by Eq. (17b) may have numerical singularity. To avoid that, the control law for $z_f \to 0$ is formulated differently. Substituting Eq. (16) into Eq. (14b), we obtain the final flight-path angle as a function of the current state:

$$\gamma_f^{(0)} = -\frac{3}{2}\left(\frac{\delta_f - \delta + \gamma z_f}{z_f}\right) + \gamma \tag{19}$$

Because $\lim_{z_f \to 0}\gamma_f^{(0)} = \gamma$, the term in the brackets in Eq. (19) approaches zero as $z_f \to 0$ and thus can be expanded about $z_f = 0$ by using a power series with respect to $z_f$. Thus

$$\left(\frac{\delta_f - \delta + \gamma z_f}{z_f}\right) = \sum_{n=1}^{\infty} D_n z_f^n \tag{20}$$

By solving for $\delta_f - \delta$, substituting into the $z_f$ polynomial (18), and equating terms of equal powers of $z_f$, we obtain the coefficients $D_n$:

$$D_1 = 0, \qquad D_2 = -A_2^2/6\gamma, \qquad D_n = (3/2\gamma)D_{n-2}^2$$

Finally, keeping only the first term of the series (20) and substituting into Eq. (17b), we get an approximation to the control law that is valid for $z_f \to 0$ and is free of singularities:

$$\lim_{z_f \to 0}\eta_0^{\mathrm{opt}}(\gamma, \delta) = A_2 z_f/2\gamma \tag{21}$$

Thus the zero-order control law contains a small discontinuity when switching from Eq. (17b) to Eq. (21). However, as will be shown in the numerical example, this small discontinuity has a negligible effect on the response of the system.

*Guidance to an Optimal Cruise Altitude*

The boundary conditions for this maneuver are the initial conditions $u_i$, $\gamma_i$, $\delta_i$ and the terminal constraints

$$\gamma_f = 0 \tag{22a}$$

$$-A_2 + \varepsilon\frac{1 - e^{-2u_f}}{\delta_f} = 0 \tag{22b}$$

Note that the full-order equation of motion (not the zero-order approximation) is used to satisfy the second terminal constraint because a realistic representation of the cruise condition requires the presence of the gravity term in the equation.

For the present application, we assume that the velocity is close to the circular orbital velocity, i.e., the value of $u$ is small. Equation (22b) is then approximated by

$$A_2\delta_f - 2\varepsilon u_f = 0 \tag{23}$$

Following the methodology of Ref. 11, we use the terminal constraints (22a) and (23) to form an augmented cost function

$$\bar{J} = -u_f + \nu_1(A_2\delta_f - 2\varepsilon u_f) + \nu_2\gamma_f \tag{24}$$

where $\nu_1$ and $\nu_2$ are constants. The boundary conditions of the Lagrange multipliers are determined by differentiating Eq. (24) with respect to the terminal states:

$$\lambda_{0_{u_f}} = -1 - 2\varepsilon\nu_1, \qquad \lambda_{0_{\gamma_f}} = \nu_2, \qquad \lambda_{0_{\delta_f}} = \nu_1 A_2 \tag{25}$$

From the Euler–Lagrange equations (12), we get $\lambda_{0_\delta} = \nu_1 A_2$ and $\lambda_{0_u} = -1 - 2\varepsilon\nu_1$.

Equations (13) are now integrated to yield the optimal zero-order trajectory

$$u^{(0)}(z) = -A_1 z - \frac{A_2^2}{4A_1\lambda_{0_u}^2}\left\{\left[(z-z_f)^3 + z_f^3\right]\frac{\nu_1^2 A_2^2}{3} + \left[(z-z_f)^2 - z_f^2\right]\nu_1\nu_2 A_2 + \nu_2^2 z\right\} + u_i \tag{26a}$$

$$\gamma^{(0)}(z) = \frac{A_2^2}{2A_1\lambda_{0_u}}\left[\frac{\nu_1 A_2}{2}z^2 + (\nu_2 - \nu_1 A_2 z_f)z\right] + \gamma_i \tag{26b}$$

$$\delta^{(0)}(z) = -\frac{A_2^2}{2A_1\lambda_{0_u}}\left[\frac{\nu_1 A_2}{6}z^3 + (\nu_2 - \nu_1 A_2 z_f)\frac{z^2}{2}\right] - \gamma_i z + \delta_i \tag{26c}$$

$$\lambda_{0_\gamma} = \nu_1 A_2(z - z_f) + \nu_2 \tag{26d}$$

The three unknowns $\nu_1$, $\nu_2$, and $z_f$ are obtained by using the two terminal constraints (22a) and (23), together with the condition $H = 0$. Substituting $z = z_f$ in Eq. (26) and then substituting the terminal states in the constraints $\gamma_f$ and $H_f = 0$, we get two linear equations for the constants $\nu_1$ and $\nu_2$, which are readily solved to yield

$$\nu_1 = \frac{4A_1[z_f - (\gamma_i/A_2)]}{A_2^2 z_f^2 - 8A_1\varepsilon[z_f - (\gamma_i/A_2)]}$$
$$\nu_2 = \frac{2A_1A_2 z_f^2}{A_2^2 z_f^2 - 8A_1\varepsilon[z_f - (\gamma_i/A_2)]} \tag{27}$$

Equations (26) and (27) form the open-loop optimal trajectory, i.e., a trajectory that is expressed as a function of the independent variable $z$ and the initial states. As in the preceding case, we desired a closed-loop formulation of the control law. The optimal feedback control law $\eta_0^{\text{opt}}(u, \gamma, \delta)$ can now be obtained from Eq. (12) by substituting the expressions for $\lambda_{0_\gamma}(z)$ from Eq. (26d) and $\lambda_{0_u}$ from Eq. (25). To get a feedback guidance law as a function of the current states, we substitute $z = 0$ and omit the index $i$:

$$\eta_0^{\text{opt}}(u, \gamma, \delta) = 1 - 2(\gamma/A_2 z_f) \tag{28}$$

The term $z_f$ is a function of the current states and is found from the remainder terminal condition (23) by using Eqs. (26a), (26c), and (27). The result is a third-order polynomial for $z_f$:

$$z_f^3 + C_2 z_f^2 + C_1 z_f + C_0 = 0, \qquad C_2 = \frac{2\gamma}{A_2} - \frac{16A_1\varepsilon}{A_2^2}$$
$$C_1 = -\frac{6\delta}{A_1} + \frac{12\varepsilon[u + 2A_1\gamma/(3A_2)]}{A_2^2} \tag{29}$$
$$C_0 = -\frac{16A_1\varepsilon\gamma^2}{A_2^4}$$

The term $z_f$ is the solution of Eq. (29) and is a function of the current state $(u, \gamma, \delta)$; $z_f$ is determined numerically by resolving Eq. (29) for each point as we move along the trajectory, using the same single-step Newton–Raphson iteration as in the preceding case. Because the coefficient $C_0$ is always negative, the existence of at least one positive real root of Eq. (29) is assured (see further discussion in the Appendix).

As the trajectory approaches the terminal point, $z_f$ becomes small. Investigation of the polynomial (29) for small $z_f$ shows that

$$z_f \to \left(4A_1/A_2^2\right)\varepsilon + \mathcal{O}(\varepsilon^2)$$

The terminal constraints of Eqs. (22a) and (23) approach the values

$$\gamma_f \to (4A_1/A_2)\varepsilon + \mathcal{O}(\varepsilon^2)$$
$$A_2\delta_f - 2\varepsilon u_f \to -\left(8A_1^2/A_2^2\right)\varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

The terminal constraints and the terminal $z_f$ do not approach exactly zero due to the fact that, in the zero-order approximation, the equations of motion (9) do not contain gravity and inertial terms, whereas the terminal constraint (23), by definition, contains these terms. Because both $z_f$ and $\gamma_f$ approach asymptotic nonzero values with the same order of magnitude $(\varepsilon)$, there is no numerical difficulty in solving the controller formula (28) close to the terminal point. Using the asymptotic values for $z_f$ and $\gamma_f$ in Eq. (28), we obtain

$$\eta_0^{\text{opt}}(u_f, \gamma_f, \delta_f) \to 1 - 2\frac{(4A_1/A_2)\varepsilon + \mathcal{O}(\varepsilon^2)}{A_2\left[\left(4A_1/A_2^2\right)\varepsilon + \mathcal{O}(\varepsilon^2)\right]} = -1 + \mathcal{O}(\varepsilon)$$

Thus the terminal control approaches the desired value of $-1$ within the accuracy of $\varepsilon$.

**First-Order Solution**

From the series expansion (8), it follows that the states, as well as the Lagrange multipliers $\lambda_u$, $\lambda_\gamma$, and $\lambda_\delta$, can be expressed as a series expansion in $\varepsilon$. Substituting the zero- and first-order terms of the series

$$\eta = \sum_{i=0}^{\infty} \eta_i\varepsilon^i, \qquad \lambda_u = \sum_{i=0}^{\infty} \lambda_{u_i}\varepsilon^i, \qquad \lambda_\gamma = \sum_{i=0}^{\infty} \lambda_{\gamma_i}\varepsilon^i$$

into the Euler–Lagrange equations (4) and the optimality condition (6), and equating coefficients of $\varepsilon$, we get the first-order term of the controller:

$$\eta_1 = \eta_0\left[\left(\lambda_{1_\gamma}/\lambda_{0_\gamma}\right) - \left(\lambda_{1_u}/\lambda_{0_u}\right)\right] \tag{30}$$

The zero-order terms $\eta_0$, $\lambda_{0_u}$, and $\lambda_{0_\gamma}$ are determined by the already found zero-order solution and are functions of the state variables.

To derive equations for the higher-order terms, the optimization problem is formulated using the Hamilton–Jacoby–Bellman equation[11]:

$$\frac{\partial P}{\partial z} = -H^{\text{opt}} \tag{31}$$

where $P$ is the optimal return function, which is the cost function as determined using the optimal controller. The partials of $P$ with respect to the states are the Lagrange multipliers $\lambda_u$, $\lambda_\gamma$, and $\lambda_\delta$.

The optimal return function $P(u, \gamma, \delta, z)$ is also expressed as the series expansion

$$P(u, \gamma, \delta, z) = \sum_{i=0}^{\infty} P_i(u, \gamma, \delta, z)\varepsilon^i$$

Substituting into the Hamilton–Jacoby–Bellman equation (31), and using the general matrix formulation (3) for the equations of motion, we get

$$\sum_{i=0}^{\infty}\frac{\partial P_i}{\partial z}\varepsilon^i = -\frac{\partial P}{\partial \boldsymbol{x}}[\boldsymbol{f}(\boldsymbol{x}, \eta^{\text{opt}}) + \boldsymbol{g}(\boldsymbol{x})\varepsilon]$$
$$= -\left(\sum_{i=0}^{\infty}\frac{\partial P_i}{\partial \boldsymbol{x}}\varepsilon^i\right)\left[\boldsymbol{f}\left(\boldsymbol{x}, \sum_{i=0}^{\infty}\eta_i^{\text{opt}}\varepsilon^i\right) + \boldsymbol{g}(\boldsymbol{x})\varepsilon\right] \tag{32}$$

The vector function

$$f\left(x, \sum_{i=0}^{\infty} \eta_i^{\text{opt}} \varepsilon^i\right)$$

which is the right-hand side of the zero-order equations of motion (2), is expanded as a power series of $\varepsilon$:

$$f\left(x, \sum_{i=0}^{\infty} \eta_i^{\text{opt}} \varepsilon^i\right) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i f\left(x, \eta_0^{\text{opt}}\right)}{\partial \eta^i} \left(\sum_{j=1}^{\infty} \eta_j^{\text{opt}} \varepsilon^j\right)^i$$

$$\equiv \sum_{i=0}^{\infty} f_i \varepsilon^i$$

The first-order term of the series is then $f_1 = [\partial f(x, \eta_0)/\partial \eta]\eta_1^{\text{opt}}$. By expanding Eq. (32) and then equating coefficients of $\varepsilon^i$, we obtain a set of partial differential equations for the terms $P_i$ of the series expansion of the optimal return function.

The first equation, which is obtained by equating coefficients of $\varepsilon$, is

$$\frac{\partial P_1}{\partial z} + \frac{\partial P_1}{\partial x} f_0^{\text{opt}}\left[x^{(0)}, \eta_0^{\text{opt}}\right] = -\frac{\partial P_0}{\partial x} g\left[x^{(0)}\right] - \frac{\partial P_0}{\partial x} f_1 \qquad (33)$$

Substituting $f_1 = [\partial f(x, \eta_0^{\text{opt}})/\partial \eta]\eta_1^{\text{opt}}$ and remembering that from the optimality condition of the zero-order controller we have $H_\eta = (\partial P_0/\partial x)[\partial f(x, \eta_0^{\text{opt}})/\partial \eta] = 0$, the second term in the right-hand side of Eq. (33) vanishes. Then

$$\frac{\partial P_1}{\partial z} + \frac{\partial P_1}{\partial x} f_0^{\text{opt}}\left[x^{(0)}, \eta_0^{\text{opt}}\right] = -\frac{\partial P_0}{\partial x} g\left[x^{(0)}\right] \qquad (34)$$

The term $x^{(0)}$ is the state vector $[u^{(0)}, \gamma^{(0)}, \delta^{(0)}]^T$, and the vector $f_0^{\text{opt}}[x^{(0)}, \eta_0^{\text{opt}}]$ is the right-hand side of the zero-order equations of motion [Eq. (2) with $\varepsilon = 0$], with the zero-order optimal controller. The term $g[x^{(0)}]$ is the vector that multiplies $\varepsilon$ in the equations of motion (2), evaluated for the zero-order solution. The partials $\partial P_0/\partial x = (\lambda_{0_u}, \lambda_{0_\gamma}, \lambda_{0_\delta})$ are also obtained from the zero-order solution, as derived in the preceding section.

Equation (34) is a linear partial differential equation for the function $P_1$. The coefficients of this equation, as well as the right-hand side, are functions of the already found zero-order solution. Equations of this type can be solved by the characteristics method.[12]

Because $dx^{(0)}/dz = f[x^{(0)}, \eta_0^{\text{opt}}]$, the left-hand side of Eq. (34) is a total differential. Thus the solution is determined by integrating the right-hand side along the zero-order solution, which is defined in Eqs. (14) and (17b) or Eqs. (26) and (29), depending on the maneuver type. Performing the integration and remembering that the boundary condition of Eq. (31) is $P_1[x_f^{(0)}, z_f] = 0$, we obtain

$$P_1(u, \gamma, \delta) = \int_{z=0}^{z=z_f} P_{0_x} g\left[x^{(0)}\right] dz$$

$$= \int_0^{z_f} \left[-\lambda_{0_u}(z) \frac{\gamma^{(0)}(z) e^{-2u^{(0)}(z)}}{\delta^{(0)}(z)} + \lambda_{0_\gamma}(z) \frac{1 - e^{-2u^{(0)}(z)}}{\delta^{(0)}(z)}\right] dz \qquad (35)$$

Here $P_{0_x} = \partial P_0/\partial x$; $\lambda_{0_u}$ and $\lambda_{0_\gamma}(z)$ are obtained from the zero-order solution for each maneuver type.

To get the first-order control function $\eta_1(u, \gamma, \delta)$ [Eq. (25)], the first-order terms of the Lagrange multipliers $\lambda_{1_u}$ and $\lambda_{1_\gamma}$ are determined by partial differentiation of Eq. (35):

$$\lambda_{1_\gamma}(u, \gamma, \delta) = \frac{\partial P_1}{\partial \gamma} = r_1(u, \gamma, \delta, z_f) \frac{\partial z_f}{\partial \gamma} + \int_0^{z_f} \frac{\partial r_1(u, \gamma, \delta, z)}{\partial \gamma} dz$$

$$\lambda_{1_u}(u, \gamma, \delta) = \frac{\partial P_1}{\partial u} = r_1(u, \gamma, \delta, z_f) \frac{\partial z_f}{\partial u} + \int_0^{z_f} \frac{\partial r_1(u, \gamma, \delta, z)}{\partial u} dz \qquad (36)$$

where $r_1(u, \gamma, \delta, z)$ is the expression under the integral in Eq. (35). This expression is a function of the integration variable $z$ and the current state $(u, \gamma, \delta)$, which is kept constant along the integration. All of the partials are derived analytically, using Eqs. (14), (17b), and (18) or Eqs. (26), (28), and (29), depending on the maneuver type. The integrals (36) are then solved numerically. The integration of the equations is performed at each time step, with the current state variables as the initial conditions of the integrals. The complete solution of the closed-loop control law involves the combined solution of the zero-order controller [Eqs. (14), (17b), and (18) or Eqs. (26), (28), and (29)] and the first-order controller [Eqs. (30–36)] at each time step.

## Exact Solution

To evaluate the quality of the approximate solutions, the complete two-point boundary-value problem was solved numerically for a few examples. The solution was obtained by the shooting method discussed in Ref. 11, using the zero-order solution as an initial guess. The results are discussed in the next section.

## Numerical Simulation Results

Numerical simulations are presented for a space vehicle with $m/S = 300$ kg/m$^2$, $C_{D_0} = 0.1$, and $(C_L/C_D)_{\max} = 1.67$. The simulations were performed by integrating the full equations of motion (1) with the feedback control laws for the zero- and first-order solutions. In addition, a numerical solution of the full two-point boundary-value problem was obtained for one example for each maneuver type. Discussion of the main results is presented in this section.

### Pull-Up Maneuver

The simulations were performed for the following conditions: initial velocity of 8200 m/s, initial flight-path angle of –0.02 rad, and initial altitude varying in the range 70–100 km. The final altitude was set to be equal to the initial altitude.

Time histories of the altitude and velocity that were simulated with the exact and zero- and first-order control laws for the 90-km case are shown in Figs. 2 and 3, respectively. The normalized lift coefficient time histories for each control law are plotted in Fig. 4. Table 1 summarizes the velocity loss for several trajectories with different initial altitudes for the zero- and first-order control laws.

Comparing the performance for the different control laws, we make the following observations.

1) Comparing the altitude and velocity of the approximate controllers with that of the exact optimal one, it is observed that, for the example that is discussed here, the approximate solutions are very close to the exact one. The velocity loss with the exact controller is 48 m/s, whereas the velocity losses with the approximate controllers are 49 m/s for the first-order and 51 m/s for the zero-order laws. The proximity of the altitude and velocity time histories for the various controllers, as presented in Figs. 2 and 3, demonstrates the validity of the assumptions and shows that the exact numerical solution can be avoided in favor of an approximate solution that requires much less computational effort.

2) For the final portion of the maneuver, which is performed at high altitude, the first-order control law (Fig. 4) is considerably different from both the zero-order and the exact control laws. However, this has only a minor effect on the trajectory. Evidently, at high altitude, i.e., low density, the trajectory is not too sensitive to lift variations.

3) Comparing the performance of the zero-order controller to that of the first-order controller (Table 1), it can be seen that the

**Table 1 Pull-up maneuver**

| Entry | Total maneuver time, s | | Velocity loss, m/s | |
|---|---|---|---|---|
| altitude, km | Zero order | First order | Zero order | First order |
| 100 | 323 | 337 | 20 | 19 |
| 90 | 275 | 290 | 51 | 49 |
| 80 | 193 | 213 | 100 | 91 |
| 70 | 100 | 118 | 146 | 135 |

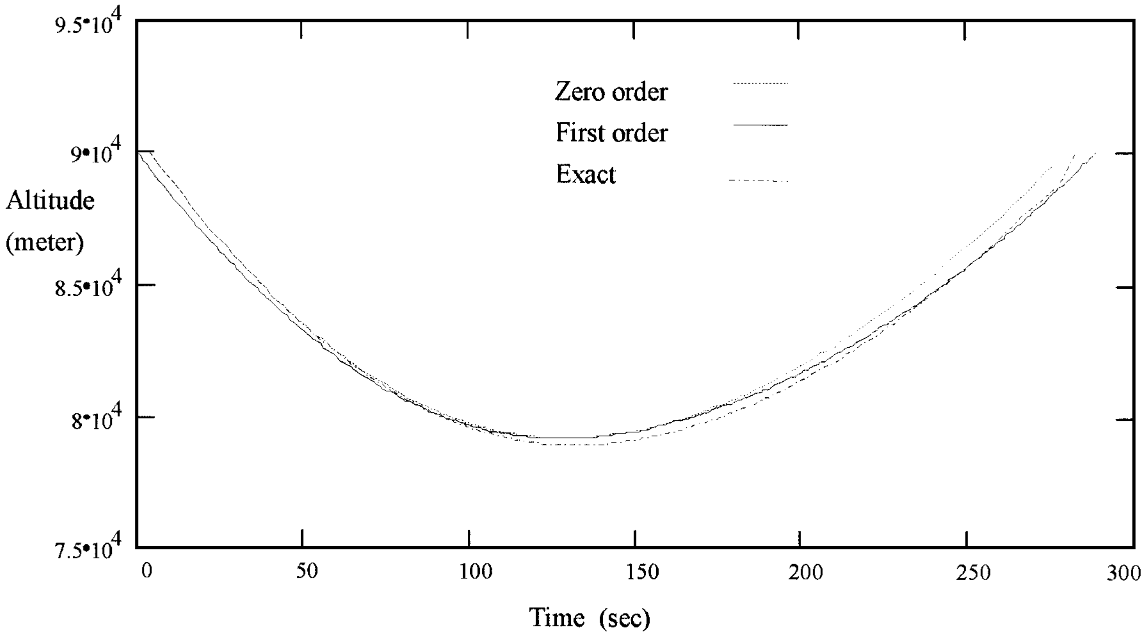


**Fig. 2 Pull-up maneuver: altitude vs time.**

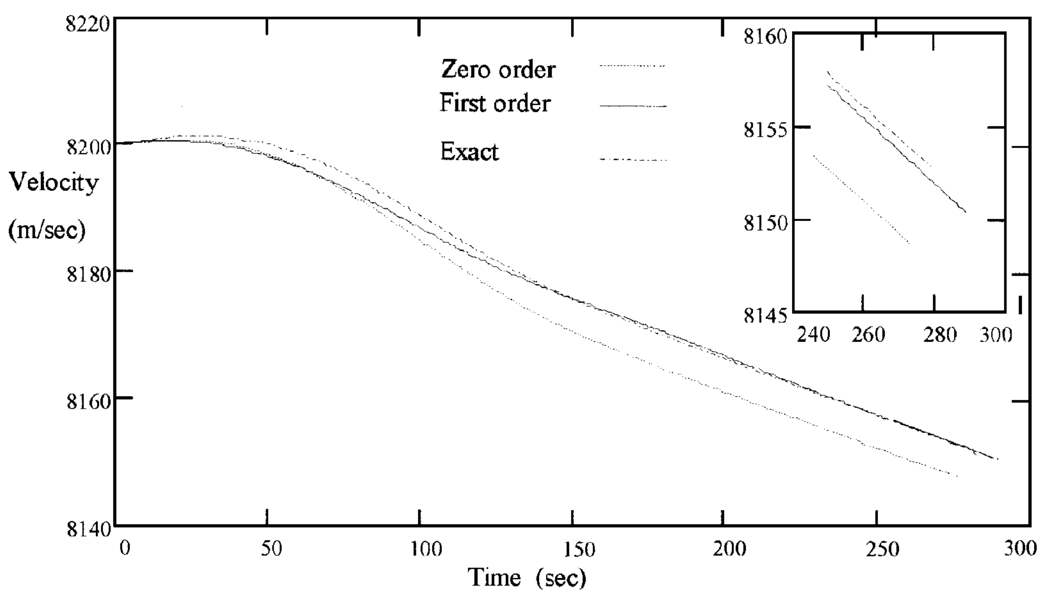


**Fig. 3 Pull-up maneuver: velocity vs time; initial altitude, 90 km.**

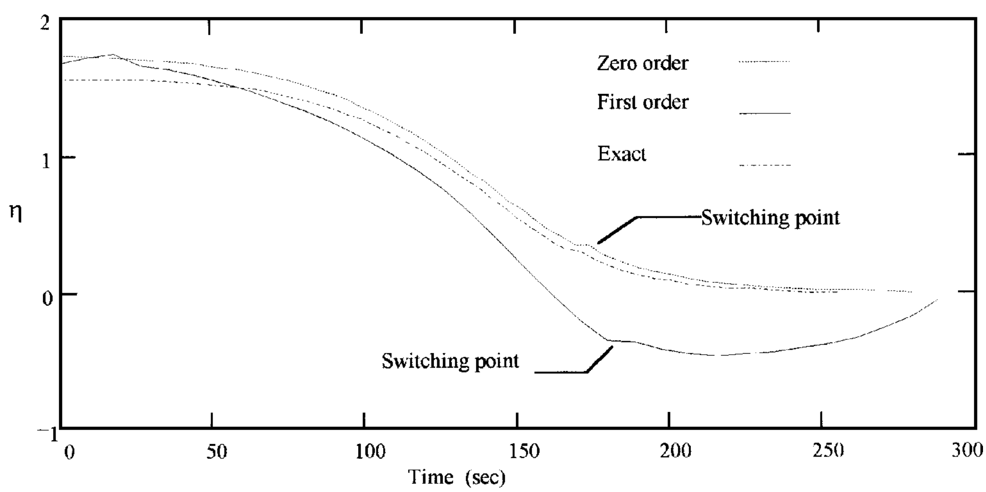


**Fig. 4 Pull-up maneuver: control law vs time; initial altitude, 90 km.**

improvement, i.e., reduction of the velocity loss, is about 5–9%. The lower the altitude, the higher the improvement.

### Guidance to an Optimal Cruise Altitude

The simulations were performed for the following conditions: initial velocity of 9000 m/s, initial flight-path angle range from −0.035 to −0.065 rad, and initial altitude range of 70–100 km.

Plots of the altitude and velocity that were simulated with the exact and zero- and first-order control laws, for initial altitude of 70 km

**Table 2 Guidance to an optimal cruise altitude; entry velocity, 9000 m/s**

| Entry altitude, km | Entry angle, rad | Velocity loss, m/s Zero order | First order |
|---|---|---|---|
| 100 | −0.065 | 253 | 244 |
| 90 | −0.054 | 260 | 242 |
| 80 | −0.045 | 295 | 292 |
| 70 | −0.035 | 264 | 261 |

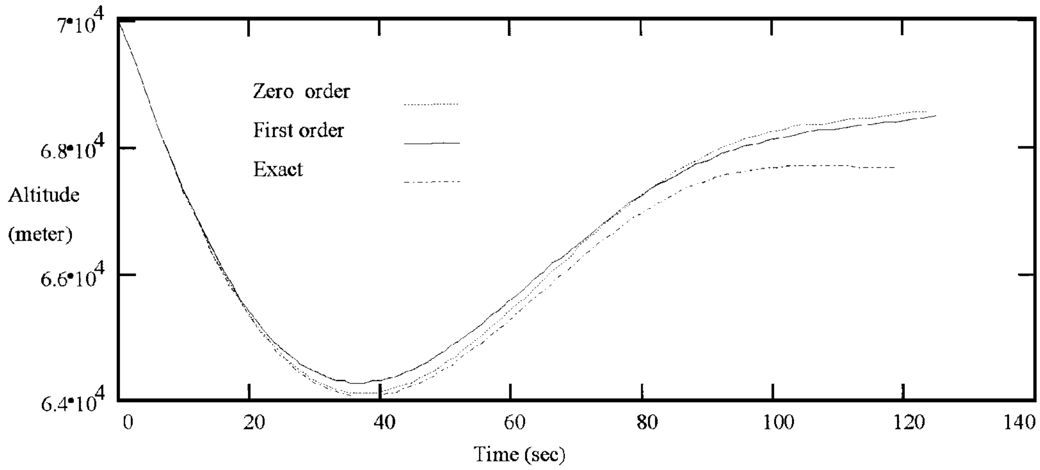


**Fig. 5 Guidance to an optimal cruise altitude: altitude vs time.**

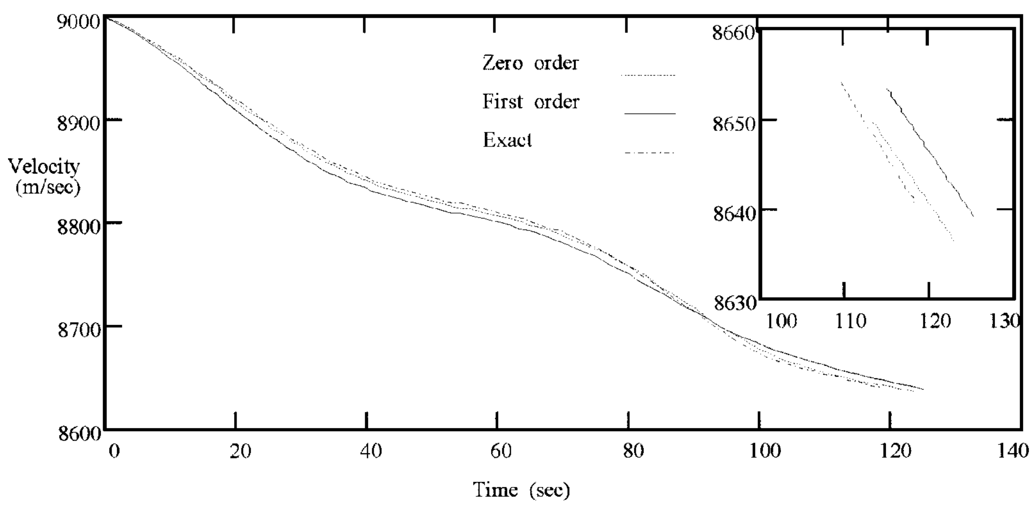


**Fig. 6 Guidance to an optimal cruise altitude: velocity vs time.**

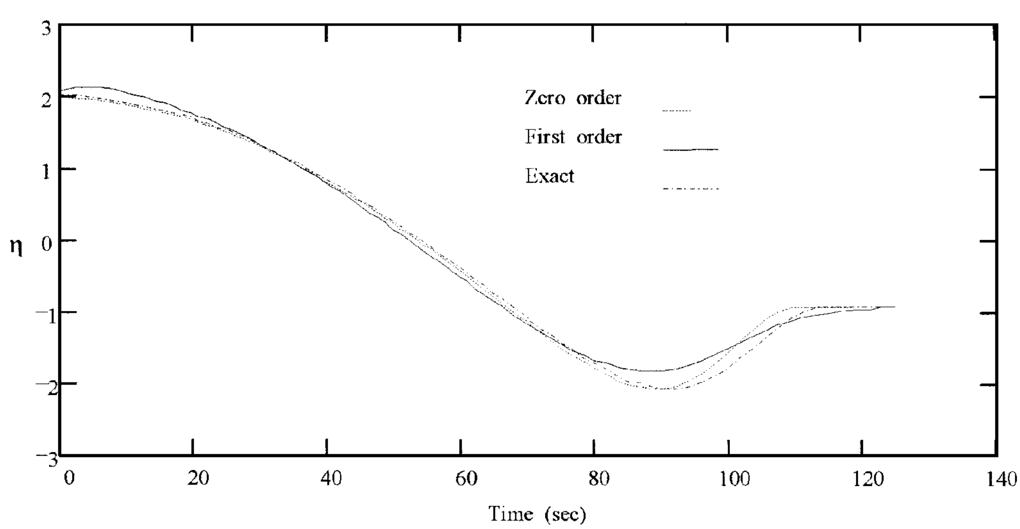


**Fig. 7 Guidance to an optimal cruise altitude: control law vs time.**

and initial flight-path angle of –0.035 rad, are shown in Figs. 5 and 6, respectively. The normalized lift coefficient time histories for each control law are plotted in Fig. 7. Table 2 summarizes the velocity loss for several trajectories with different initial altitudes for the zero- and first-order control laws.

Here we observe basically the same behavior as in the preceding maneuver: the approximate solutions are close to the exact one. There is a slight improvement with the first-order solution over the zero-order solution. It can be seen (Table 2) that the relative reduction in the velocity loss is greater for the high initial altitude case (100 km) than for the low initial altitude case. It seems that for this maneuver type, the high altitude case can be slightly improved by including the first-order terms in the control law.

## Conclusions

This paper presents the derivation of approximate optimal control laws for planar aeroassisted maneuvers that are based on the assumption that the aerodynamic forces are dominant along the atmospheric passage. The approximate control law is expressed as a series expansion with respect to a small parameter that reflects the ratio between the forces. In this paper, zero- and first-order terms of the series expansion are derived. The zero-order solution is formulated in closed form. Based on the zero-order solution, the first-order solution is derived. The resulting first-order solution requires only quadrature integration without iterations. The control laws are expressed as feedback functions of the state variables.

This methodology is applied here for two types of planar maneuvers. These maneuvers may be practical when a planar rotation of the line of apsides of an elliptic orbit is desired. For these two maneuvers, the zero-order solution can be obtained by an analytical solution of the zero-order, two-point boundary-value problem, and thus the first-order solution can be derived.

Numerical examples that simulated these maneuvers demonstrated the following points.

1) Approximate control laws have very close performance to the exact optimal solution, which was obtained by a numerical solution of the two-point boundary-value problem.

2) In most cases, the zero-order solution is adequate. The first-order solution can improve the performance by up to an additional 10% of the velocity loss.

3) The computational requirements of the approximate solution are considerably lower than for the exact solution. The approximate solutions do not require iterations and can be implemented on board.

## Appendix: Mathematical and Numerical Aspects of Calculating $z_f$

To obtain $z_f$, the polynomials (18) or (29) (depending on the maneuver type) are solved at each time step. The following issues are discussed: 1) existence of a real positive root, 2) treatment of the case of multiple positive roots, and 3) use of the single step Newton–Raphson iteration for the on-line computation.

The relevant root of the polynomial should be real and positive. By applying Descartes' rule of signs to each polynomial, we observe that the number of sign changes in the quartic (18) depends on the sign of $(\delta_f - \delta)\gamma$. For $(\delta_f - \delta)\gamma > 0$, there is one sign change, and therefore there is one real positive root. For $(\delta_f - \delta)\gamma < 0$, there are three sign changes, and therefore the polynomial may have either one or three real positive roots.

We also observe that in the cubic (29), there is at least one sign change because the coefficient $C_0$ is always negative. Therefore there is at least one positive real root.

In the case where more than one positive real root exists, the cost function $J = -u_f$ is computed for each root. The one root that minimizes the cost function is chosen. This calculation is performed only once, at the beginning of the trajectory. The values of $z_f$ for each time update along the trajectory are obtained by a single-step Newton–Raphson iteration.

Let us represent the polynomials (18) and (29) by

$$F(\boldsymbol{x}, z_f) = 0$$

where $\boldsymbol{x}$ is the current state vector $(u, \gamma, \delta)^T$. Suppose that the root of this function at time step $n$ is known. Then

$$F(\boldsymbol{x}_n, z_{f_n}) = 0$$

As we move to the next time step, we have, to first order,

$$F(\boldsymbol{x}_{n+1}, z_{f_n}) = F(\boldsymbol{x}_n, z_{f_n}) + \frac{\partial F(\boldsymbol{x}_n, z_{f_n})}{\partial \boldsymbol{x}} \Delta \boldsymbol{x}_n = \frac{\partial F(\boldsymbol{x}_n, z_{f_n})}{\partial \boldsymbol{x}} \Delta \boldsymbol{x}_n$$

The updated root $z_{f_{n+1}}$ should satisfy $F(\boldsymbol{x}_{n+1}, z_{f_{n+1}}) = 0$. Expanding to first order yields

$$F(\boldsymbol{x}_{n+1}, z_{f_{n+1}}) = F(\boldsymbol{x}_n, z_{f_n}) + \frac{\partial F(\boldsymbol{x}_n, z_{f_n})}{\partial \boldsymbol{x}} \Delta \boldsymbol{x}_n + \frac{\partial F(\boldsymbol{x}_n, z_{f_n})}{\partial z_f} \Delta z_n = 0$$

Because $F(\boldsymbol{x}_n, z_{f_n}) = 0$ and $F(\boldsymbol{x}_{n+1}, z_{f_n}) = [\partial F(\boldsymbol{x}_n, z_{f_n}) / \partial \boldsymbol{x}] \Delta x_n$, we have

$$\Delta z_{f_n} = -\frac{F(\boldsymbol{x}_{n+1}, z_{f_n})}{\partial F(\boldsymbol{x}_n, z_{f_n}) / \partial z_f}$$

hence $z_{f_{n+1}} = z_{f_n} - [F(z_{f_n}) / F'(z_{f_n})]$, which is essentially a single-step Newton–Raphson iteration.

By following this method, the value of $z_f$ is tracked along the trajectory. In case of multiple roots, care should be taken to avoid converging to the other roots. This is done by monitoring the first and second derivatives of $F(\boldsymbol{x}_n, z_{f_n})$ with respect to $z_f$. As long as those derivatives have the same sign as their original values, i.e., the slope and the curvature are the same, tracking of the correct root is assured.

To summarize, the computation procedure of $z_f$ is as follows.

1) Find all of the positive real roots of the polynomial at the beginning of the trajectory. This is an off-line operation that is done only once and is performed by using a numerical method such as Newton–Raphson. Existence of at least one positive real root is guaranteed.

2) If more than one positive real root exists, choose the one for which the cost function at the beginning of the trajectory is minimum.

3) Use the single-step Newton–Raphson iteration to update $z_f$ at each new time step, while monitoring the first- and second-order derivatives.

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